

# FAMILIES OF STRICTLY PSEUDOCONVEX DOMAINS AND PEAK FUNCTIONS

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ABSTRACT. We prove that given a family  $(G_t)$  of strictly pseudoconvex domains varying in  $C^2$  topology on domains, there exists a continuously varying family of peak functions  $h_{t,\zeta}$  for all  $G_t$  at every  $\zeta \in \partial G_t$ .

## 1. INTRODUCTION

Let  $D \subset \mathbb{C}^n$  be a bounded domain and let  $\zeta$  be a boundary point of  $D$ . It is called a *peak point* with respect to  $\mathcal{O}(\overline{D})$ , the family of functions which are holomorphic in a neighborhood of  $\overline{D}$ , if there exist a function  $f \in \mathcal{O}(\overline{D})$  such that  $f(\zeta) = 1$  and  $f(\overline{D} \setminus \{\zeta\}) \subset \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ . Such a function is a *peak function for  $D$  at  $\zeta$* . The concept of peak functions appears to be a powerful tool in complex analysis with many applications. It has been used to show the existence of (complete) proper holomorphic embeddings of strictly pseudoconvex domains into the unit ball  $\mathbb{B}^N$  with large  $N$  (see [5],[3]), to estimate the boundary behavior of Carathéodory and Kobayashi metrics ([1],[6]), or to construct the solution operators for  $\bar{\partial}$  problem with  $L^\infty$  or Hölder estimates ([4],[10]), just to name a few of those applications.

It is well known that every boundary point of strictly pseudoconvex domain is a peak point. Even more is true, in [6] it is showed that, given a strictly pseudoconvex domain  $G$ , there exists an open neighborhood  $\widehat{G}$  of  $G$ , and a continuous function  $h : \widehat{G} \times \partial G \rightarrow \mathbb{C}$  such that for  $\zeta \in \partial G$ , the function  $h(\cdot; \zeta)$  is a peak function for  $G$  at  $\zeta$ .

In a recent paper [2] the following question has been posed:

**Problem 1.1.** Let  $\rho : \mathbb{D} \times \mathbb{C}^n \rightarrow \mathbb{R}$  be a plurisubharmonic function of class  $\mathcal{C}^{2+k}$ ,  $k \in \mathbb{N} \cup \{0\}$ , such that for any  $z \in \mathbb{D}$  the truncated function  $\rho|_{\{z\} \times \mathbb{C}^n}$  is strictly plurisubharmonic. Define  $G_z := \{w \in \mathbb{C}^n : \rho(z, w) < 0\}$ ,  $z \in \mathbb{D}$ . This can be understood as a family of strictly pseudoconvex domains over  $\mathbb{D}$ . Does there exist a  $\mathcal{C}^k$ -continuously varying family  $(h_{z,\zeta})_{z \in \mathbb{D}, \zeta \in \partial G_z}$  of peak functions for  $G_z$  at  $\zeta$ ?

We answer this question affirmatively in the case  $k = 0$  and under additional assumption that, roughly speaking, the function  $\rho$  keeps its regularity up to the set  $\Omega \times \mathbb{C}^n$ , where  $\Omega$  is some open neighborhood of  $\overline{\mathbb{D}}$ . Namely, let us consider the following:

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**Situation 1.2.** Let  $(G_t)_{t \in T}$  be a family of bounded strictly pseudoconvex domains, where  $T \subset \mathbb{C}$  is a compact set. Suppose we have a domain  $U \subset \subset \mathbb{C}^n$  such that

- (1)  $\bigcup_{t \in T} \partial G_t \subset \subset U$ ,
- (2) for each  $t \in T$  there exists a defining function  $r_t$  for  $G_t$  satisfying with neighborhood  $\partial G_t \subset U$  all the conditions (A)-(F) below (see Section 2),
- (3) for any  $\varepsilon > 0$  there exists a  $\delta > 0$  such that for any  $s, t \in T$  with  $|s - t| \leq \delta$  there is  $\|r_t - r_s\|_{\mathcal{C}^2(U)} < \varepsilon$ .

Observe that the above setting is completely in the spirit of the formulation of Problem 1.1:

- (i) The assumption that all the functions  $r_t$  satisfy (A)-(F) with common neighborhood  $\partial G_t \subset U$  stays in relation with the fact that in Problem 1.1 all the defining functions for domains  $D_z$  have the same domain of definition ( $\mathbb{C}^n$ ).
- (ii) The assumption (3) comes from the fact that the function  $\rho$  in Problem 1.1 is of class at least  $\mathcal{C}^2$ .
- (iii) The compactness of the set of parameters ( $T$ ) reflects the above mentioned assumption that  $\rho$  continues to be of class  $\mathcal{C}^2$  up to  $\Omega \times \mathbb{C}^n$ , with  $\Omega$  being some neighborhood of  $\mathbb{D}$ .

We shall prove the following:

**Theorem 1.3.** *Let  $(G_t)_{t \in T}$  be a family of strictly pseudoconvex domains as in Situation 1.2. Then there exists an  $\varepsilon > 0$  such that for any  $\eta_1 < \varepsilon$  there exist an  $\eta_2 > 0$  and positive constants  $d_1, d_2$  such that for any  $t \in T$  there exist a domain  $\widehat{G}_t$  containing  $\overline{G}_t$ , and functions  $h_t(\cdot; \zeta) \in \mathcal{O}(\widehat{G}_t)$ ,  $\zeta \in \partial G_t$  fulfilling the following conditions:*

- (a)  $h_t(\zeta; \zeta) = 1, |h_t(\cdot; \zeta)| < 1$  on  $\overline{G}_t \setminus \{\zeta\}$  (in particular,  $h_t(\cdot; \zeta)$  is a peak function for  $G_t$  at  $\zeta$ ),
- (b)  $|1 - h_t(z; \zeta)| \leq d_1 \|z - \zeta\|, z \in \widehat{G}_t \cap \mathbb{B}(\zeta, \eta_2)$ ,
- (c)  $|h_t(z; \zeta)| \leq d_2 < 1, z \in \overline{G}_t, \|z - \zeta\| \geq \eta_1$ .

Moreover, the constants  $\varepsilon, \eta_2, d_1, d_2$ , domains  $\widehat{G}_t$ , and functions  $h_t(\cdot; \zeta)$  may be chosen in such a way that for any  $\alpha > 0$  and any fixed triple  $(t_0, \zeta_0, z_0)$ , where  $t_0 \in T, \zeta_0 \in \partial G_{t_0}$ , and  $z_0 \in \widehat{G}_{t_0}$ , there exists a  $\delta > 0$  such that whenever the triple  $(s, \xi, w)$  satisfies  $s \in T, \xi \in \partial G_s, w \in \widehat{G}_s$ , and  $\max\{|s - t_0|, \|\xi - \zeta_0\|, \|w - z_0\|\} < \delta$ , then  $|h_{t_0}(z_0; \zeta_0) - h_s(w; \xi)| < \alpha$ .

The latter property will be referred to as *continuity*.

**Remark 1.4.** It is known that for each  $t \in T$  there exists an  $\varepsilon = \varepsilon(t) > 0$  such that for any  $\eta_1 < \varepsilon$  there exist a positive  $\eta_2 = \eta_2(t) < \eta_1$ , constants  $d_1 = d_1(t), d_2 = d_2(t) \in \mathbb{R}$ , domain  $\widehat{G}_t$  containing  $\overline{G}_t$ , and functions  $h_t(\cdot; \zeta) \in \mathcal{O}(\widehat{G}_t), \zeta \in \partial G_t$  satisfying (a)-(c). This is a subject of Theorem 19.1.2 from

[7]. The strength of our result dwells in the fact that all the constants  $\varepsilon, \eta_2, d_1, d_2$  are chosen independently of  $t$  and in the continuity property.

In Section 2 we recall some preliminaries concerning the strictly pseudoconvex domains. The proof of Theorem 1.3 is presented in Section 3.

## 2. STRICTLY PSEUDOCONVEX DOMAINS

Let  $D \subset\subset \mathbb{C}^n$  be a domain. It is called a *strictly pseudoconvex* if there exist a neighborhood  $U$  of  $\partial D$  and a *defining function*  $r : U \rightarrow \mathbb{R}$  of class  $\mathcal{C}^2$  and such that

- (A)  $D \cap U = \{z \in U : r(z) < 0\}$ ,
- (B)  $(\mathbb{C}^n \setminus \overline{D}) \cap U = \{z \in U : r(z) > 0\}$ ,
- (C)  $\nabla r(z) \neq 0$  for  $z \in \partial D$ , where  $\nabla r(z) := \left( \frac{\partial r}{\partial z_1}(z), \dots, \frac{\partial r}{\partial z_n}(z) \right)$ ,

together with

$$\mathcal{L}_r(z; X) > 0 \text{ for } z \in \partial D \text{ and nonzero } X \in T_z^{\mathbb{C}}(\partial D),$$

where  $\mathcal{L}_r$  denotes the Levi form of  $r$  and  $T_z^{\mathbb{C}}(\partial D)$  is the complex tangent space to  $\partial D$  at  $z$ .

It is known that  $U$  and  $r$  can be chosen to satisfy (A)-(C) and, additionally:

- (D)  $\mathcal{L}_r(z; X) > 0$  for  $z \in U$  and all nonzero  $X \in \mathbb{C}^n$ ,
- (E)  $\|\nabla r(z)\| = 1, z \in \partial D$ ,
- (F) for every  $z \in D \cap U$  there is a unique  $\pi(z) \in \partial D$  with

$$\text{dist}(z, \partial D) = \|z - \pi(z)\|,$$

cf. [8],[9]. Note that for a function  $r$  as above and a point  $\zeta \in \partial G$ , Taylor expansion of  $r$  at  $\zeta$  has the following form:

$$(2.1) \quad r(z) = r(\zeta) - 2\text{Re}P(z; \zeta) + \mathcal{L}_r(\zeta; z - \zeta) + o(\|z - \zeta\|^2),$$

where

$$P(z; \zeta) := - \sum_{j=1}^n \frac{\partial r}{\partial z_j}(\zeta)(z_j - \zeta_j) - \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 r}{\partial z_i \partial z_j}(\zeta)(z_i - \zeta_i)(z_j - \zeta_j)$$

is the *Levi polynomial* of  $r$  at  $\zeta$ .

## 3. PROOF OF THEOREM 1.3

We divide the proof into two parts. First we give the construction of  $\widehat{G}_t$  and  $h_t(\cdot; \zeta), t \in T$ , and define the constants  $\varepsilon, \eta_2, d_1$ , and  $d_2$ , all independent of  $t$ . This is refinement of the construction from the proof of Theorem 19.1.2 from [7]. Note that in order to get the independence of all the constants from  $t$ , we must be more careful here. In the second part we prove the continuity property.

*Construction of  $\widehat{G}_t$  and  $h_t(\cdot; \zeta)$  and the choice of  $\varepsilon, \eta_2, d_1$ , and  $d_2$ .* For  $t \in T$  and  $\zeta \in \partial G_t$  let  $P_t(z; \zeta)$  be the Levi polynomial of  $r_t$  at  $\zeta$ .

Fix an  $\varepsilon_1 > 0$  such that  $U' := \bigcup_{t \in T, \zeta \in \partial G_t} \mathbb{B}(\zeta, \varepsilon_1) \subset\subset U$ .

There exists a constant  $C_1 = C_1(t) < 1$  such that

$$\mathcal{L}_{r_t}(z; X) \geq C_1 \|X\|^2, \quad z \in U', X \in \mathbb{C}^n.$$

Indeed,  $\mathcal{L}_{r_t}$  is continuous and positive on  $U \times (\mathbb{C}^n \setminus \{0\})$ , so it attains its minimum  $C_1(t) > 0$  on  $\overline{U'} \times \mathbb{S}^{n-1}$ . Since for any nonzero  $X \in \mathbb{C}^n$  we have  $\frac{X}{\|X\|} \in \mathbb{S}^{n-1}$ , we get the required inequality. Moreover, from the assumption (3) it follows that for  $s$  from some neighborhood of  $t$  we have

$$\mathcal{L}_{r_s}(z; X) \geq \frac{C_1(t)}{2} \|X\|^2, \quad z \in U', X \in \mathbb{C}^n.$$

The compactness argument then gives that  $C_1$  may be chosen independently of  $t$ .

Taylor formula (2.1) yields that with some  $0 < C_2 < C_1$  there is

$$(3.1) \quad r_t(z) \geq -2\operatorname{Re}P_t(z; \zeta) + C_2 \|z - \zeta\|^2$$

for  $\|z - \zeta\| < \varepsilon_2(t) < \varepsilon_1, \zeta \in \partial G_t$ , where  $\varepsilon_2(t)$  is independent of  $\zeta \in \partial G_t$  (and even of  $\zeta \in W \subset\subset U$ , some neighborhood of  $\partial G_t$  - see [11], Proposition II.2.16). Moreover, from the proof of Theorem V.3.6 from [11] it follows that for  $s$  close enough to  $t$  we have

$$r_s(z) \geq r_s(\zeta) - 2\operatorname{Re}P_s(z; \zeta) + \frac{C_2}{2} \|z - \zeta\|^2, \quad \zeta \in W, \|z - \zeta\| < \varepsilon_2(t).$$

Therefore, for  $s$  near to  $t$ , and for  $\xi \in \partial G_s$ , the following estimate holds true:

$$r_s(z) \geq -2\operatorname{Re}P_s(z; \xi) + \frac{C_2}{2} \|z - \xi\|^2, \quad \|z - \xi\| < \varepsilon_2(t).$$

The compactness argument then implies that  $C_2$  and  $\varepsilon_2$  in (3.1) may be chosen independently of  $t$ .

Let  $0 < \eta_1 < \varepsilon_2$  and  $\hat{\chi} \in \mathcal{C}^\infty(\mathbb{R}, [0, 1])$  be such that  $\hat{\chi}(t) = 1$  for  $t \leq \frac{\eta_1}{2}$  and  $\hat{\chi}(t) = 0$  for  $t \geq \eta_1$ . Put  $\chi(z; \zeta) := \hat{\chi}(\|z - \zeta\|)$ . This is a smooth function on  $\mathbb{C}^n \times \mathbb{C}^n$ , taking its values in  $[0, 1]$ .

Define

$$\varphi_t(z; \zeta) := \chi(z; \zeta)P_t(z; \zeta) + (1 - \chi(z; \zeta))\|z - \zeta\|^2, \quad z \in \mathbb{C}^n.$$

Observe that if  $\|z - \zeta\| \leq \frac{\eta_1}{2}$ , then  $\varphi_t(z; \zeta) = P_t(z; \zeta)$ . In particular  $\varphi_t(\cdot; \zeta) \in \mathcal{O}(\mathbb{B}(\zeta, \frac{\eta_1}{2}))$ . Furthermore, for  $z$  satisfying  $\|z - \zeta\| \geq \frac{\eta_1}{2}$  and  $r_t(z) < C_2 \frac{\eta_1^2}{8}$  the following estimate holds true:

$$(3.2) \quad 2\operatorname{Re}\varphi_t(z; \zeta) \geq C_2 \frac{\eta_1^2}{8} > 0.$$

Take  $0 < \eta_t < C_2 \frac{\eta_1^2}{8}$  such that the connected component  $\widetilde{G}_t$  containing  $\overline{G}_t$  of the open set

$$G_t \cup \{z \in U' : r_t(z) < \eta_t\}$$

is a strictly pseudoconvex domain, relatively compact in  $G_t \cup U'$ . Because of the assumption (3), there exists a positive number  $\beta$  such that for  $s$  close to  $t$  the connected component  $\widetilde{G}_s$  containing  $\overline{G}_s$  of the set

$$G_s \cup \{z \in U' : r_s(z) < \eta_t - \beta\}$$

is a strictly pseudoconvex domain, relatively compact in  $G_s \cup U'$ . Making again use of the compactness of  $T$ , we conclude that in fact  $\eta = \eta_t$  may be taken independently of  $t$ . Note that for the family  $(\widetilde{G}_t)_{t \in T}$  the assumption (3) remains true.

The function  $\varphi_t(\cdot; \zeta) \in \mathcal{C}^\infty(\mathbb{C}^n)$  does not vanish on  $\widetilde{G} \setminus \mathbb{B}(\zeta, \frac{\eta_1}{2})$  and is in  $\mathcal{O}(\mathbb{B}(\zeta, \frac{\eta_1}{2}))$ . Therefore  $\bar{\partial} \frac{1}{\varphi_t(\cdot; \zeta)}$  defines a  $\bar{\partial}$ -closed  $\mathcal{C}^\infty$  form

$$\alpha_t(\cdot; \zeta) = \sum_{j=1}^n \alpha_{t,j}(\cdot; \zeta) dz_j$$

on  $\widetilde{G}_t$ , where

$$\alpha_{t,j} = \begin{cases} 0, & z \in \widetilde{G}_t \cap \mathbb{B}(\zeta, \frac{\eta_1}{2}), \\ -\frac{\partial \varphi_t}{\partial z_j}(z; \zeta) \cdot \frac{1}{\varphi_t^2(z; \zeta)}, & z \in \widetilde{G}_t \setminus \mathbb{B}(\zeta, \frac{\eta_1}{2}). \end{cases}$$

Thanks to (3.2) we have  $\|\alpha_{t,j}(\cdot; \zeta)\|_{\widetilde{G}_t} \leq C_3$ , where, utilizing the compactness of  $T$  together with the assumption (3), we deliver that  $C_3$  is independent of  $t$  and  $\zeta \in \partial G_t$ . [11, Theorem V.2.7] gives then the functions  $v_t(\cdot; \zeta) \in \mathcal{C}^\infty(\widetilde{G}_t)$  with  $\bar{\partial} v_t(\cdot; \zeta) = \alpha_t(\cdot; \zeta)$  and

$$\|v_t(\cdot; \zeta)\|_{\widetilde{G}_t} \leq C_4,$$

where  $C_4$  does not depend on  $\zeta \in \partial G_t$ . Moreover, by [11, Theorem V.3.6] and the compactness of  $T$ ,  $C_4$  may be chosen to be independent of  $t$ .

Define

$$f_t(\cdot; \zeta) := \frac{1}{\varphi_t}(\cdot; \zeta) + C_4 - v_t(\cdot; \zeta), \quad z \in \widetilde{G}_t \setminus Z_t(\zeta),$$

where

$$Z_t(\zeta) := \{z \in \widetilde{G}_t : \varphi_t(z; \zeta) = 0\}.$$

Then  $f_t(\cdot; \zeta) \in \mathcal{O}(\widetilde{G}_t \setminus Z_t(\zeta))$  as well as

$$\operatorname{Re} f_t(\cdot; \zeta) > 0$$

on the set  $(\widetilde{G}_t \setminus \mathbb{B}(\zeta, \frac{\eta_1}{2})) \cup (\overline{G}_t \setminus \{\zeta\})$ , in virtue of (3.1) and (3.2). Since for any  $\zeta \neq z_0 \in \partial G_t \cap \mathbb{B}(\zeta, \frac{\eta_1}{2})$  there exists a neighborhood  $U_{z_0}$  of  $z_0$  such that  $\operatorname{Re} f_t(\cdot; \zeta) > 0$  on  $U_{z_0}$ , we conclude that there exists a neighborhood  $U_{t,\zeta}$  of  $\overline{G}_t \setminus \{\zeta\}$  such that the function

$$h_t(\cdot; \zeta) := \exp(-g_t(\cdot; \zeta)),$$

where  $g_t(\cdot; \zeta) := \frac{1}{f_t(\cdot; \zeta)}$ , is holomorphic on  $H_{t,\zeta} := (\widetilde{G}_t \setminus \mathbb{B}(\zeta, \frac{\eta_1}{2})) \cup U_{t,\zeta}$ . Note that  $h_t$  takes its values in  $\mathbb{D}$ .

There exists a  $C_5 > 0$ , independent on  $t$ , such that

$$|P_t(z; \zeta)| \leq C_5 \|z - \zeta\|, \quad \zeta, z \in U'.$$

Therefore, since for  $0 < \eta_2 < \min \left\{ \frac{\eta_1}{2}, \frac{1}{4C_4C_5} \right\}$ , which now is independent of  $t$ , and for  $z \in ((H_{t,\zeta} \cup \mathbb{B}(\zeta, \eta_2)) \cap \mathbb{B}(\zeta, \eta_2)) \setminus Z_t(\zeta)$  the following equality holds true:

$$g_t(z; \zeta) = \frac{P_t(z; \zeta)}{1 - P_t(z; \zeta)(v_t(z; \zeta) - C_4)},$$

we conclude that  $g_t(\cdot; \zeta)$  is bounded near  $Z_t(\zeta)$ , which yields it extends to be holomorphic on  $\widehat{H}_{t,\zeta} := H_{t,\zeta} \cup \mathbb{B}(\zeta, \eta_2)$ .

Now  $\widehat{H}_{t,\zeta}$  depends on  $\zeta$ , but using the inclusion  $\overline{G_t} \subset \widehat{H}_{t,\zeta}$ , we may find some  $\widehat{G_t}$ , strictly pseudoconvex domain which is independent on  $\zeta \in \partial G_t$ , such that  $\overline{G_t} \subset \widehat{G_t} \subset \widehat{H}_{t,\zeta}$  for each  $\zeta \in \partial G_t$ , and with the property that  $h_t(\cdot; \zeta) \in \mathcal{O}(\widehat{G_t})$ ,  $\zeta \in \partial G_t$  (use the joint continuity of  $\varphi_t$  with respect to  $z$  and  $\zeta$  to shrink  $\widehat{H}_{t,\zeta}$  little bit to get some domain with desired properties, independent on  $\xi$  close to  $\zeta$ , and finally apply the compactness of  $\partial G_t$ ). Let  $C_6$ , independent on  $t$  and  $\zeta \in \partial G_t$ , such that for  $z \in \widehat{G_t}$  with  $\|z - \zeta\| < \eta_2$  we have

$$g_t(z; \zeta) \leq \frac{C_5 \|z - \zeta\|}{1 - 2C_4C_5 \|z - \zeta\|} \leq C_6 \|z - \zeta\|.$$

This implies

$$|1 - h_t(z; \zeta)| \leq C_7 |g_t(z; \zeta)| \leq C_6 C_7 \|z - \zeta\| =: d_1 \|z - \zeta\|$$

for  $z \in \widehat{G_t}$ ,  $\|z - \zeta\| < \eta_2$ ,  $\zeta \in \partial G_t$ , if only  $C_7$  is chosen so that

$$|e^\lambda - 1| \leq C_7 |\lambda|, \quad |\lambda| \leq C_6 \eta_2.$$

In particular,  $d_1$  does not depend on  $t$  and we have  $h_t(\zeta; \zeta) = 1$ .

Furthermore, for  $z \in \overline{G_t}$ ,  $\|z - \zeta\| \geq \eta_1$  there is

$$\begin{aligned} \text{Reg}_t(z; \zeta) &= \|z - \zeta\|^2 \frac{1 + \|z - \zeta\|^2 (C_4 - \text{Rev}_t(z; \zeta))}{|1 - \|z - \zeta\|^2 (v_t(z; \zeta) - C_4)|^2} \\ &\geq \frac{\eta_1^2}{(1 + 2(\text{diam} U)^2 C_4)^2} =: C_8, \end{aligned}$$

which gives

$$(3.3) \quad |h_t(z; \zeta)| \leq e^{-C_8} =: d_2 < 1.$$

Observe that  $d_2$  is independent on  $t$ . □

*Proof of continuity.* Fix  $\alpha > 0$ ,  $t_0 \in T$ ,  $\zeta_0 \in \partial G_{t_0}$ , and  $z_0 \in \widehat{G_{t_0}}$ . Let  $K_0$  be a compact subset of  $\widehat{G_{t_0}}$ , containing in its interior the set  $\overline{G_{t_0}} \cup \{z_0\}$ . In the sequel we shall use the following convention: whenever we say that the triple  $(s, \xi, w)$  is near to  $(t_0, \zeta_0, z_0)$ , it will carry the additional information that  $\xi \in \partial G_s$ ,  $w \in \widehat{G_s}$ , unless explicitly stated otherwise.

Observe that for  $(s, \xi)$  close to  $(t_0, \zeta_0)$  (even without requiring that  $\xi \in \partial G_s$ ), and any  $z, w \in U'$  we have

$$|P_{t_0}(z; \zeta_0) - P_s(w; \xi)| < M_1 \alpha$$

with some positive  $M_1$ . In particular, the same estimate is true for  $z = z_0$  and  $w$  close to  $z_0$ .

Further, using the fact that all the functions  $\varphi_t$  are continuous as functions of both variables, we conclude that for  $(s, \xi)$  close to  $(t_0, \zeta_0)$  we have

$$\|\varphi_{t_0}(\cdot; \zeta_0) - \varphi_s(\cdot; \xi)\|_{U'} < M_2 \alpha$$

with some positive  $M_2$ .

For  $(s, \xi)$  near  $(t_0, \zeta_0)$  we have

$$\left\| \frac{\partial \varphi_{t_0}}{\partial \bar{z}_j}(\cdot; \zeta_0) - \frac{\partial \varphi_s}{\partial \bar{z}_j}(\cdot; \xi) \right\|_{U'} < M_3 \alpha$$

with some positive  $M_3$ . Furthermore, for  $z \in \widetilde{G}_s \cap \widetilde{G}_t$  the following estimates hold true:

(I) If  $z \notin \mathbb{B}(\zeta_0, \frac{\eta_1}{2}) \cup \mathbb{B}(\xi, \frac{\eta_1}{2})$ , then

$$|\alpha_{t_0, j}(z; \zeta_0) - \alpha_{s, j}(z; \xi)| < L \alpha,$$

where positive constant  $L$  does not depend on  $z$  as above. Indeed,

$$\begin{aligned} \left| \frac{\frac{\partial \varphi_{t_0}}{\partial \bar{z}_j}(z; \zeta_0)}{\varphi_{t_0}^2(z; \zeta_0)} - \frac{\frac{\partial \varphi_s}{\partial \bar{z}_j}(z; \xi)}{\varphi_s^2(z; \xi)} \right| &= \left| \frac{\varphi_s^2(z; \xi) \frac{\partial \varphi_{t_0}}{\partial \bar{z}_j}(z; \zeta_0) - \varphi_{t_0}^2(z; \zeta_0) \frac{\partial \varphi_s}{\partial \bar{z}_j}(z; \xi)}{\varphi_{t_0}^2(z; \zeta_0) \varphi_s^2(z; \xi)} \right| \\ &\leq \frac{64}{C_2^2 \eta_1^4} \left| \varphi_s^2(z; \xi) \frac{\partial \varphi_{t_0}}{\partial \bar{z}_j}(z; \zeta_0) - \varphi_{t_0}^2(z; \zeta_0) \frac{\partial \varphi_s}{\partial \bar{z}_j}(z; \xi) \right| \\ &\leq \frac{64}{C_2^2 \eta_1^4} \|\varphi_s^2\|_{U'} \left\| \frac{\partial \varphi_{t_0}}{\partial \bar{z}_j}(\cdot; \zeta_0) - \frac{\partial \varphi_s}{\partial \bar{z}_j}(\cdot; \xi) \right\|_{U'} + \left\| \frac{\partial \varphi_s}{\partial \bar{z}_j}(\cdot; \xi) \right\|_{U'} \|\varphi_s^2 - \varphi_{t_0}^2\| \\ &\leq \frac{64}{C_2^2 \eta_1^4} L_1 M_3 \alpha + L_2 M_2 \alpha =: L \alpha, \end{aligned}$$

where the first inequality is the consequence of (3.2).

(II) If  $z \in \mathbb{B}(\zeta_0, \frac{\eta_1}{2}) \cup \mathbb{B}(\xi, \frac{\eta_1}{2})$ :

Observe that letting  $\xi$  close to  $\zeta_0$ , we may make the balls arbitrarily close each other. Using then the assumption (3), the fact that  $\eta$  were chosen to be strictly smaller than  $C_2 \frac{\eta_1^2}{8}$ , and the strictness of uniform estimate (3.2), we see that for  $(s, \xi)$  close enough to  $(t_0, \zeta_0)$  the estimate similar to the previous one holds true for  $z \in S := \bigcup_{w: \|w - \zeta_0\| = \frac{\eta_1}{2}} \mathbb{B}(w, \gamma)$  with some sufficiently small

$\gamma > 0$ , (and is independent on such  $z$ ). Additionally,  $(s, \xi)$  may be chosen so that  $S' := (\mathbb{B}(\zeta_0, \frac{\eta_1}{2}) \cup \mathbb{B}(\xi, \frac{\eta_1}{2})) \setminus S \subset \mathbb{B}(\zeta_0, \frac{\eta_1}{2}) \cap \mathbb{B}(\xi, \frac{\eta_1}{2})$ .

Noting that for  $z \in S'$  and  $(s, \xi)$  as above  $\alpha_{t_0, j}(z; \zeta_0) = \alpha_{s, j}(z; \xi) = 0$ , we conclude that

$$\|\alpha_{t_0}(\cdot; \zeta_0) - \alpha_s(\cdot; \xi)\|_{\widetilde{G_{t_0}} \cap \widetilde{G_s}} \leq M_4 \alpha$$

with some positive  $M_4$ .

Ofcourse  $\overline{G_{t_0}} \subset \widetilde{G_{t_0}}$ . This yields that for  $s$  close to  $t_0$  we have  $\overline{G_{t_0}} \subset \widetilde{G_s}$  as well as  $\overline{G_s} \subset \widetilde{G_{t_0}}$  (the assumption (3) remains true for the family  $(\widetilde{G_t})_{t \in T}$ ). For  $s$  close to  $t_0$  we may now pick some  $G_{t_0, s}$ , a strictly pseudoconvex domain with smooth boundary and such that

$$\overline{G_s} \cup \overline{G_{t_0}} \subset K_0 \subset \subset G_{t_0, s} \subset \subset \widetilde{G_s} \cap \widetilde{G_{t_0}}.$$

Again thanks to the property (3),  $G_{t_0, s}$  may be chosen independently of  $s$  if  $s$  is close enough to  $t_0$ . For such  $s$ , denote it by  $G^{t_0}$ . Then, using Lemma 2 from [6], we find some positive constant  $\Gamma$  such that

$$\|v_{t_0}(\cdot; \zeta_0) - v_s(\cdot; \xi)\|_{K_0} \leq \Gamma \|\alpha_{t_0}(\cdot; \zeta_0) - \alpha_s(\cdot; \xi)\|_{G^{t_0}} \leq \Gamma M_4 \alpha =: M_5 \alpha.$$

Consequently, for  $(s, \xi, w)$  close to  $(t_0, \zeta_0, z_0)$  there is

$$|v_{t_0}(z_0; \zeta_0) - v_s(w; \xi)| \leq |v_{t_0}(z_0; \zeta_0) - v_{t_0}(w; \zeta_0)| + |v_{t_0}(w; \zeta_0) - v_s(w; \xi)| \leq M_6 \alpha$$

for some positive  $M_6$  (use the smoothness of  $v_{t_0}(\cdot; \zeta_0)$ ).

There are two cases to be considered:

*Case 1.*  $z_0 \in H_{t_0, \zeta_0} \cap \text{int} K_0$ .

Then  $\varphi_{t_0}(z_0; \zeta_0) \neq 0$  and for  $(s, \xi, w)$  near  $(t_0, \zeta_0, z_0)$  we have  $\varphi_s(w; \xi) \neq 0$ . For such  $(s, \xi, w)$  we have

$$\begin{aligned} |f_{t_0}(z_0; \zeta_0) - f_s(w; \xi)| &\leq \left| \frac{1}{\varphi_{t_0}(z_0; \zeta_0)} - \frac{1}{\varphi_s(w; \xi)} \right| + |v_{t_0}(z_0; \zeta_0) - v_s(w; \xi)| \\ &\leq \left| \frac{\varphi_s(w; \xi) - \varphi_{t_0}(z_0; \zeta_0)}{\varphi_{t_0}(z_0; \zeta_0) \varphi_s(w; \xi)} \right| + M_6 \alpha. \end{aligned}$$

Considering the last but one term, its denominator is bounded below by some positive constant for  $(s, \xi, w)$  close to  $(t_0, \zeta_0, z_0)$ , and the counter is estimated from above by  $M_2 \alpha$ . Thus for  $(s, \xi, w)$  close to  $(t_0, \zeta_0, z_0)$

$$|f_{t_0}(z_0; \zeta_0) - f_s(w; \xi)| \leq M_7 \alpha$$

for some positive  $M_7$ .

In our situation the function  $g_{t_0}(\cdot; \zeta_0)$  is holomorphic in a neighborhood of  $z_0$  and so is  $g_s(\cdot; \xi)$  for  $(s, \xi)$  close to  $(t_0, \zeta_0)$ . We conclude that for  $(s, \xi, w)$  close to  $(t_0, \zeta_0, z_0)$  there is

$$|g_{t_0}(z_0; \zeta_0) - g_s(w; \xi)| \leq M_8 \alpha$$

for some positive  $M_8$ , and

$$|h_{t_0}(z_0; \zeta_0) - h_s(w; \xi)| = \left| \exp(-g_{t_0}(z_0; \zeta_0)) - \exp(-g_s(w; \xi)) \right| \leq M_9 \alpha$$



for some positive  $M_9$ .

*Case 2.*  $z_0 \in (\widetilde{G_{t_0}} \cap \mathbb{B}(\zeta_0, \eta_2)) \cap \text{int} K_{t_0}$ .

(I) Suppose  $\varphi_{t_0}(z_0; \zeta_0) \neq 0$ .

It is equivalent to  $P_{t_0}(z_0; \zeta_0) \neq 0$ . This yields that  $P_s(w; \xi) \neq 0$  for  $(s, \xi, w)$  close to  $(t_0, \zeta_0, z_0)$ . Then

$$\begin{aligned} & |g_{t_0}(z_0; \zeta_0) - g_s(w; \xi)| \\ &= \left| \frac{P_{t_0}(z_0; \zeta_0)}{1 - P_{t_0}(z_0; \zeta_0)(v_{t_0}(z_0; \zeta_0) - C_4)} - \frac{P_s(w; \xi)}{1 - P_s(w; \xi)(v_s(w; \xi) - C_4)} \right| \\ &\leq N|P_{t_0}(z_0; \zeta_0) - P_s(w; \xi)| + N|P_{t_0}(z_0; \zeta_0)P_s(w; \xi)| |v_{t_0}(z_0; \zeta_0) - v_s(w; \xi)| \\ &\leq NM_1\alpha + NM_6\alpha =: M_{10}\alpha, \end{aligned}$$

and similarly as in the previous case

$$|h_{t_0}(z_0; \zeta_0) - h_s(w; \xi)| \leq M_{11}\alpha$$

with some positive  $N, M_{10}$ , and  $M_{11}$ .

(II) Suppose  $\varphi_{t_0}(z_0; \zeta_0) = 0$ .

This is equivalent to  $P_{t_0}(z_0; \zeta_0) = 0$ . Then for some positive  $\rho$  we have  $\mathbb{B}(z_0, \rho) \subset\subset K_0 \cap \mathbb{B}(\zeta_0, \eta_2)$ . Similarly, for  $(s, \xi)$  close to  $(t_0, \zeta_0)$  there is  $\mathbb{B}(z_0, \rho) \subset\subset K_0 \cap \mathbb{B}(\xi, \eta_2)$ . Therefore, because of the choice of  $d_2$  in (3.3), for  $(s, \xi, w)$  close to  $(t_0, \zeta_0, z_0)$ ,  $w \in \mathbb{B}(z_0, \rho/2)$  there is

$$\begin{aligned} |h_{t_0}(w; \zeta_0) - h_s(w; \xi)| &\leq |1 - h_{t_0}(w; \zeta_0)| + |1 - h_s(w; \xi)| \\ &\leq d_1(\|w - \zeta_0\| + \|w - \xi\|) \leq 2d_1\eta_2. \end{aligned}$$

Consequently, since the functions  $h_{t_0}(\cdot; \zeta_0) - h_s(\cdot; \xi)$  are holomorphic in suitable neighborhood of  $z_0$  for  $(s, \xi)$  close to  $(t_0, \zeta_0)$ , for some positive  $\tilde{\rho} < \rho/2$ , for every  $x, y \in \mathbb{B}(z_0, \tilde{\rho}/2)$  we have

$$(3.4) \quad |h_{t_0}(x; \zeta_0) - h_s(x; \xi) - h_{t_0}(y; \zeta_0) + h_s(y; \xi)| \leq \alpha.$$

Moreover,  $\tilde{\rho}$  may be chosen so that for  $v, w \in \mathbb{B}(z_0, \tilde{\rho}/2)$  there is

$$(3.5) \quad |h_{t_0}(v; \zeta_0) - h_{t_0}(w; \zeta_0)| \leq \alpha,$$

by continuity of  $h_{t_0}(\cdot; \zeta_0)$ .

Fix some  $w_0 \in \mathbb{B}(z_0, \tilde{\rho}/2)$  such that  $P_{t_0}(w_0; \zeta_0) \neq 0$ . Then for  $(s, \xi)$  near  $(t_0, \zeta_0)$ , by virtue of the subcase (I), we have

$$|h_{t_0}(w_0; \zeta_0) - h_s(w_0; \xi)| \leq \alpha.$$

Finally, for  $w \in \mathbb{B}(z_0, \tilde{\rho}/2)$  and  $(s, \xi)$  close to  $(t_0, \zeta_0)$  we have

$$\begin{aligned} |h_{t_0}(z_0; \zeta_0) - h_s(w; \xi)| &\leq |h_{t_0}(z_0; \zeta_0) - h_{t_0}(w_0; \zeta_0)| + |h_{t_0}(w_0; \zeta_0) - h_s(w_0; \xi)| \\ &\quad + |h_s(w_0; \xi) - h_s(w; \xi)| \leq \alpha + \alpha + 2\alpha = 4\alpha, \end{aligned}$$

where the last estimate follows from (3.4) and (3.5), which leads us to the conclusion.  $\square$

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